The Algebraic Structure of Unity of Roots

Zhelin Qi*

Hangzhou foreign language school, 309 Liuhe Road, Xihu District, Hangzhou City, Zhejiang Province, Hangzhou, China, 310023

*Corresponding author: 1685002178@qq.com

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Abstract: Calculus over complex variables is the subject of complex function theory. In this work, we take a systematic look at what we know about complex numbers from a higher perspective. The collection of all complex numbers is called a field in algebra. All complex numbers, geometrically, constitute a full metric space with an attractive topological structure. Mathematicians use complex numbers to solve technical issues in a very succinct manner. Complex numbers' computational and algebraic characteristics are summarized, as well as their geometric representation. Then we go over the group structure of unity of roots in depth and provide proofs for a number of beautiful conclusions.

1. Introduction

Because of its importance in the quantum gate, qubit, phasor, potential flow, and quantum physics, the topic of complex numbers must be introduced [1-10]. Complex numbers are important in many fields of mathematics, including number theory and probability theory. Mathematicians use complex numbers to solve technical issues in a very succinct manner. This paper addresses the geometric representation of complex numbers as well as its computational and algebraic characteristics. Then we go into the group structure of unity of roots in depth and provide comprehensive proofs for a number of stunning conclusions. We provide thorough proof of the following identity.

$$(1+2\cos\frac{2\pi}{n})(1+2\cos\frac{4\pi}{n})(1+2\cos\frac{6\pi}{n})\ldots(1+2\cos\frac{2k\pi}{n})=3 \tag{1}$$

Where n is prime.

2. Main works

2.1 Complex numbers

A complex number is an element in the form of

$$x+iy \tag{2}$$

Where x, y are two arbitrary real numbers, and i is the root of the equation$x^2+1=0$.

We use the notation C as the set of all of the complex numbers, i.e.

$$\mathrm{C} =: \{x+i\cdot y|x\in R, y\in R\} \tag{3}$$

2. For a given complex number $z = x + i\cdot y$, where x, y are two real numbers.

We define its real part as x and use the notation $Re(z)$ to represent its real part.; we say y is its imaginary part and use the notation $Im(z)$ to represent its imaginary part; we define its modulus as

$$|z|=\sqrt{(x^2+y^2)} \tag{4}$$

Given two complex numbers, we denote them asz_1, z_2. That is, z_1, $z_2 \in C$. Then we have that $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$.

$$\begin{aligned} \text{Proof. } |z_1+z_2|^2 &= (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\ &= z_1\cdot\overline{z_1}+z_2\cdot\overline{z_2}+z_1\cdot\overline{x_2}+\overline{z_1}\cdot z_2 \\ &= |z_1|^2+|z_2|^2+(z_1\cdot\overline{z_2}+\overline{z_1}\cdot z_2). \end{aligned} \tag{5}$$

$$\begin{aligned} \text{Notice that } |z_1-z_2|^2 &= (z_1-z_2)\ (\overline{z_1-z_2}) \\ &= (z_1-z_2)\ (\overline{z_1}-\overline{z_2}) \\ &= z_1\cdot\overline{z_1}+z_2\cdot\overline{x_2}-z_1\cdot\overline{z_2}-\overline{z_1}\cdot z_2 \\ &= |z_1|^2+|z_2|^2-(z_1\cdot\overline{z_2}+\overline{z_1}\cdot z_2). \end{aligned} \tag{6}$$

Thus, the left-hand side $=\left(|z_1|^2+|z_2|^2+(z_1\cdot\overline{z_2}+\overline{z_1}\cdot z_2)\right)+\left(|z_1|^2+|z_2|^2-(z_1\cdot\overline{x_2}+\overline{z_1}\cdot z_2)\right) = 2(|z_1|^2+|z_2|^2) =$ the left-hand side.

2.2 Complex plane

After forming a rectangular coordinate -system, the real number pair mean a point. So, $\mathrm{Z} = \mathrm{a} + \mathrm{bi}$ means abscissa is a and ordinate is b. And if that point's polar coordinates be (r, θ), so the complex number $\mathrm{z} = \mathrm{a} + \mathrm{bi}$ can show as z=r (cos$\theta + isin\theta$). R = |z| =$\sqrt{a^2+b^2}$ is the modulus of z which identify before θ called by the auxiliary angle, so $\theta + 2k\pi$ will also be auxiliary angle of z, the k will be any inter number, so that the auxiliary angle will be infinity. But in$Argz$, there is only one θ follow by the conditions which is -$\pi < \theta \leq \theta$, so called this θ be the principal value of the auxiliary angle of z and write asArgz. So Argz $=$ argz $+ 2\mathrm{k}\pi, \mathrm{k} \in$ Z..

Here, Z means the whole group of the integer number and the auxiliary angle of 0 has no meaning.

Then let the complex number $\mathrm{z} = \mathrm{a} + \mathrm{bi}$ be the projection on x-axis and y-axis be the vector of a and b respectively, and at that time, it can be found the complex number and vector as the same meaning. Easily, find from a start point and the end point of a vector, are the complex number $z_1$and $z_2$respectively, so complex number this vector show is $z_1 - z_2$, so $|z_1 - z_2|$ shows the distance between the z_1 and z_2. In particular, when the starting point of a vector is the origin, the complex numbers represented by its terminus are the same as those represented by the vector. It can be seen from this that It follows that the addition of complex numbers defined earlier is the same as the addition of vectors: take the starting points of two non-zero vectors z_1 and z_2 which do not coincide, at the origin, with Z1 and z_2 as two side, draw a parallelogram for both sides, then the vector along the diagonal from the origin point will represent$z_1 + z_2$; The vector starting with z_2 and ending with z_1 represents z_1-z_2. Inequality $|z_1 + z_2|$ ≤$|z_1|$+$|z_2|$, it is actually the simplest geometric proposition that the sum of the two sides of a triangle is greater than the third side. To illustrate the geometric meaning of complex multiplication, by using the trigonometric representation of complex numbers, suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \tag{7}$$

$$z2 = r_2(\cos\theta_2 + i\sin\theta_2 \tag{8}$$

So, $z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$.

From this, $|z_1z_2| = |z_1| + |z_2|, \mathrm{Arg}(z_1z_2) = \mathrm{Arg}z_1 + \mathrm{Arg}z_2$. the first equation was proved in Section 2.1The second equation should be understood as the equality of two sets. That is to say, the product of two complex numbers is a complex number whose magnitude is the product of the magnitudes of two complex numbers, whose arguments are the sum of the arguments of the two complex numbers. Geometrically, multiplying the complex number z by 0 is equivalent to rotating z counterclockwise by an Angle of magnitude argw and extending the length of z by |w|, In particular, if w is the unit vector, then w multiplied by z results in turning z counterclockwise by an angle of magnitudeargw. For example, given i is the unit vector, and its argument$\frac{\pi}{2}$, so iz is the vector obtained by turning z counterclockwise by$\frac{\pi}{2}$. This geometric intuition is very useful in thinking about problems. For the division of the complex number, because of$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$, it follows that$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, Arg$\left(\frac{z_1}{z_2}\right) = Argz_1 - Argz_2$. Here, the second equation is also understood to be equal to the set, which means that the Angle between vector z_1 and z_2 is $\text{Arg}\left(\frac{z_1}{z_2}\right)$ to represent this simple fact in discussing some geometry problem. For example, it is easy to prove that the necessary and sufficient condition for the perpendicularity of vectors (z_1 and z_2 is Re ($z_1\overline{z_2}$) = 0. This is because the included angle between (z_1 and z_2 is$\pm\frac{\pi}{2}$, namely$arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$, which shows that ($z_1$ is a pure imaginary number, so z1z2 is also a pure imaginary number, that is, Re ($z_1\overline{z_2}$) = 0. Similarly, the necessary and sufficient condition for (z_1 and z_2 to be parallel is $\text{Im}(z_1\overline{z_2})$= 0.

It is sometimes very convenient to use knowledge of complex numbers to deal with geometric problems. Here are two examples. In the triangle below, AB=AC, PQ=RS, M and N are the midpoints of PR and QS respectively: MN⊥BC.

Prove: If A is taken as the coordinate origin and the straight line where AB is located is taken as the X axis, then the coordinates of P and Q are a and a + h respectively. If $e^{i\theta}$ is denoted as$cos\theta + i\sin\theta$, then R and S points can be represented by the complex number r$e^{i\theta}$ and (r + h)$e^{i\theta}$ respectively. Since m and N are the midpoint of PR and sq respectively, m and N can be expressed in complex numbers as:

M: $\frac{1}{2}\left(a + re^{i\theta}\right)$.N: $\frac{1}{2}\left[(a+h) + (r+h)e^{i\theta}\right]$.If write down $z_1 = \overrightarrow{MN}$, so

$$z_1 = \frac{1}{2}\left[(a+h) + (r+h)e^{i\theta}\right] - \frac{1}{2}\left(a + re^{i\theta}\right) \tag{9}$$

$$= \frac{h}{2}\left(1 + e^{i\theta}\right)$$

If write down the coordinates of B is b, because of AB=AC, so the coordinates of C is$be^{i\theta}$. If write down$z_2 = \overrightarrow{BC}$, so$z_2 = b^{i\theta} - b = b\left(e^{i\theta} - 1\right)$.

$$\text{Now, } z_1\overline{z_2} = \frac{h}{2}\left(1 + e^{i\theta}\right)b\left(e^{i\theta} - 1\right) = \frac{bh}{2}\left(e^{-i\theta} - e^{i\theta}\right) = -ibh\sin\theta \tag{10}$$

So Re ($z_1\overline{z_2}$) = 0. So $z_1 \perp z_2$, which means MN⊥BC.

For future discussion, there will be introduction of a new number ∞ into C. The modulus of this number is∞, and the argument is meaningless. The operation rule of this number and other numbers is as follows:

$$z \pm \infty = \infty, z \bullet \infty = \infty (z \neq 0) \tag{11}$$

$$\frac{z}{\infty} = 0,\ \frac{z}{0} = \infty (z \neq 0) \tag{12}$$

$0 \bullet \infty$ And $\infty \pm \infty$ do not specify their meanings, and the complex number of ∞ is introduced asC_∞, which is$C_\infty = C \cup \{\infty\}$. On the complex plane, there is no point that corresponds to∞, but imagine that there is an infinity that corresponds to∞, and the complex plane with infinity plus is called the extended plane or the closed plane, and the complex plane without infinity is also called the open plane. In the complex plane, where infinity is not the same as ordinary points, Riemann first introduced the sphere surface of complex numbers representation, in which ∞ is no difference between ordinary complex numbers, let S be the unit sphere of R^3, i.e.$S = \{(x_1, x_2, x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Equating C to a plane:$C = \{(x_1, x_2, 0) : x_1, x_2 \in R\}$.

For the north plate N of fixed S, that is, N= (0, 0, 1), for any point z on C, the line connecting N and 2 must intersect S at point P. If $|z|$ >1, P is in the northern hemisphere. If $|z|$ <1, P is in the southern hemisphere; If $|z|$=1, then P is x. It is easy to see that as z approaches∞, the corresponding point P on the sphere approaches the North Pole N. Naturally, C_∞ is refer to corresponds to the North Pole of n. Thus, all points in C_∞ (including infinity) are transplanted to the sphere, where N and other points are peer dependent.

The corresponding expression is now given. If z=x + iy, it is easy to calculate the coordinate of the intersection of zN and sphere S.$x_1 = \frac{2x}{x^2+y^2+1}, x_2 = \frac{2y}{x^2+y^2+1}, x_3 = \frac{x^2+y^2-1}{x^2+y^2+1}$.

Let's just use the complex number z, can show as: $x_1=\frac{z+\bar{z}}{1+|z|^2}, x_2=\frac{z-\bar{z}}{i(1+|z|^2)}, x_3=\frac{|z|^2-1}{|z|^2+1}$.

In this way, the coordinates of its corresponding points on the spherical soil can be calculated from x, and conversely, its corresponding points on the plane x can be calculated from the points on the sphere(x_1,x_2,x_3). In fact, get from above expression.

$$\begin{cases} x_1+ix_2=\frac{2z}{1+|z|^2} \\ 1-x_3=\frac{2}{1+|z|^2}, \end{cases} \tag{13}$$

Thus obtained,

$$z=\frac{x_1+ix_2}{1-x_3} \tag{14}$$

This is the required calculation formula.

Assume the integer n is a prime that is bigger than three, then $(1+2\cos\frac{2\pi}{n})(1+2\cos\frac{4\pi}{n})(1+2\cos\frac{6\pi}{n})\dots(1+2\cos\frac{2k\pi}{n})=3$.

Proof:

For convenience, assume that $w=e^{\frac{2\pi i}{n}}$ Then$w^n=1, w^{-\frac{n}{2}}=e^{-ni}=-1, 2\cos\frac{2k\pi}{n}=w^k+w^{-k}$.

$\prod_{k=1}^{n}(1+2\cos\frac{2k\pi}{n})=\prod_{k=1}^{n}(1+w^k+w^{-k})=\prod_{k=1}^{n}w^{-k}(w^{2k}+w^k+1)=w^{-\frac{n(n+1)}{2}}\cdot 3\prod_{k=1}^{n-1}\frac{1-w^{3k}}{1-w^k}=(-1)^{n+1}\cdot 3\prod_{k=1}^{n-1}\frac{1-w^{3k}}{1-w^k}$. The number n is a prime bigger than 3.

Thus, it must be an odd number. It follows that n+1 is an even number. Thus$(-1)^{n+1}=1$.

As a result, $\prod_{k=1}^{n}(1+2\cos\frac{2k\pi}{n})=(-1)^{n+1}.3\prod_{k=1}^{n-1}\frac{1-w^{3k}}{1-w^k}=3\prod_{k=1}^{n-1}\frac{1-w^{3k}}{1-w^k}$.

It suffices to prove that $\prod_{k=1}^{n-1}\frac{1-w^{3k}}{1-w^k}$=1.

Take the set S=$\{1,w^1,w^2,\dots,w^{n-1}\}$ into consideration under multiplication.

Assume that j∈{0,1,2,...,n-1}. $w^iw^j=w^{i+j}$. If i+j≤n-1, w^{i+j} ∈S.

If i+j>n-1, since i+j-n≤n-1, $w^{i+j}=w^{i+j-n}$ ∈A. Therefore, A is closed.

Let i,j,k∈{0,1,2,...,n-1}. $(w^iw^j)w^k=w^{i+j+k}=w^i(w^jw^k)$, so S is associative.

Let i∈{0,1,2,...,n-1}. $w^i\cdot1=1\cdot w^i=w^i$, so 1 is the identity.

Let i∈{0,1,2,...,n-1}. $w^i\cdot w^{-i}=w^{-i}\cdot w^i=1$, so w^{-i} is the inverse ofw^i.

Therefore, the set S=$\{1,w^1,w^2,\dots,w^{n-1}\}$ is a group under multiplication.

Since A= $\{w^n|n\in Z\}$, A is a cyclic group. Consider the set B=$\{1,w^3,w^6,\dots,w^{3(n-1)}\}$ under multiplication. Let i, j∈{0,3,6,...,3(n-1)}. $w^iw^j=w^{i+j}$. If i+j≤3(n-1), w^{i+j} ∈B. If i+j>3(n-1), since i+j-3n≤3(n-1), $w^{i+j}=w^{i+j-3n}$ ∈B. Therefore, B is closed.

1. Let i, j, k∈{0,3,6,...,3(n-1)}. $(w^iw^j)w^k=w^{i+j+k}=w^i(w^jw^k)$, so B is associative.
2. Let i∈{0,3,6,...,3(n-1)}. $w^i\cdot1=1\cdot w^i=w^i$, so 1 is the identity.
3. Let i∈{0,3,6,...,3(n-1)}. $w^i\cdot w^{-i}=w^{-i}\cdot w^i=1$, so w^{-i} is the inverse ofw^i.

Therefore, the set B=$\{1,w^3,w^6,\dots,w^{3(n-1)}\}$ is a group under multiplication.

Since B= $\{w^{3n}|n\in Z\}$, B is a cyclic group. Since n is a prime greater than 3, 0 is not a possible remainder of 3, 3×2… 3(n-1), divided by n. Since there are n-1 elements in the set {3, 3×2,…, 3(n-1)} and the remainders can only be 1,2,...,n-1, then the set of the remainders is {1,2,...,n-1}.Hence, the groups {3, 3×2,…, 3(n-1)} and {1,2,...,n-1} are equal under modulo of n. Notice that under the operation modulo of n, the group $\{1,w^1,w^2,\dots,w^{n-1}\}$ is isomorphic to {1,2,...,n-1}, and $\{1,w^3,w^6,\dots,w^{3(n-1)}\}$ is isomorphic to {3, 3×2,…, 3(n-1)} , so $\{1,w^1,w^2,\dots,w^{n-1}\}$ is isomorphic to $\{1,w^3,w^6,\dots,w^{3(n-1)}\}$. Because $w^n=1$, then $w^k=w^k$ mod n. Thus $\{1,w^1,w^2,\dots,w^{n-1}\}=\{1,w^3,w^6,\dots,w^{3(n-1)}\}$. ∴ $\{w^1,w^2,\dots,w^{n-1}\}=\{w^3,w^6,\dots,w^{3(n-1)}\}$. ∴

$\prod_{k=1}^{n-1} \frac{1-w^{3k}}{1-w^k} = \frac{(1-w^3)(1-w^6)\ldots(1-w^{3(n-1)})}{(1-w)(1-w^2)\ldots(1-w^{n-1})} = 1$. $\therefore (1 + 2\cos\frac{2\pi}{n})(1 + 2\cos\frac{4\pi}{n})(1 + 2\cos\frac{6\pi}{n}) \ \ldots \ (1 + 2\cos\frac{2k\pi}{n}) = 3.$

3. Conclusion

We present numerous beautiful conclusions on the roots of unity in this work, which covers the basic geometric and algebraic characteristics of complex numbers. We'll write more about whole functions, which are a type of significant holomorphic functions, in the future. We'll go through various proofs of Jensen's formula in depth, as well as growth order and infinite products. Understanding how these theorems and ideas are linked to show some interesting features of the whole function is important. Weierstrass and Hadamard's work should be taken into account in the future.

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